

## THE ASSUMPTIONS UNDERLYING THE GENERALIZED MATCHING LAW

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Allen (1981) derived the power-function generalization of the matching law from a functional equation involving relative response rates on three concurrently available schedules of reinforcement. This paper defines the conditions (relative homogeneity and independence) under which a more general class of behavioral laws reduces to the power law. The proof also removes two deficiencies of Allen's result (discussed by Houston, 1982), which are, first, that his derivation produces a power law without a bias coefficient, and second, that it holds only for experiments with three or more concurrent schedules.

*Key words:* generalized matching, power law, functional equation

The generalized version of the matching law (Baum, 1974, 1979; Staddon, 1968) states that relative rate of responding on two alternatives,  $i$  and  $j$ , is a power function of the relative rate of reinforcement obtained from the two alternatives:

$$\frac{B_i}{B_j} = c \left( \frac{R_i}{R_j} \right)^s. \quad (1)$$

The reinforcers are typically delivered by variable-interval (VI) schedules, but other schedules have been used as well. The parameter  $c$  measures bias, that is, excess preference for one alternative, and parameter  $s$  measures the degree to which response ratios are sensitive to reinforcement ratios. The case when  $c$  and  $s$  are both equal to one is of special significance because it corresponds to Herrnstein's original version of the matching law (Herrnstein, 1961, 1970):

$$\frac{B_i}{B_j} = \frac{R_i}{R_j}. \quad (2)$$

Allen (1981) has provided a formal derivation of the power law in the following way. He assumed, first, the existence of a general functional relationship between relative response rates and relative reinforcement rates:

$$\frac{B_i}{B_j} = f \left( \frac{R_i}{R_j} \right). \quad (3)$$

He then showed that if there are at least three distinct alternatives,  $i, j, k$ , and if the same functional relationship  $f$  holds for all possible alternative pairs (and all rates of reinforcement), then  $f$  has to be a power function:  $f(x) = x^s$ . Allen interpreted this derivation as a formal proof that the power function is a necessary consequence of consistent deviations from simple matching (Equation 2). In a rejoinder to Allen's article, Houston (1982) pointed out that Allen's interpretation of his result is not quite true because the notion of a consistent deviation from matching itself lacks *formal* justification. Houston also mentioned a second unsatisfactory aspect of the proof, namely that it yields a power function without a bias parameter. Allen's (1982) response that the bias parameter can be reintroduced by applying the power law to value-scaled reinforcement ratios,

$$\frac{B_i}{B_j} = f \left( \frac{V_i}{V_j} \right),$$

where  $V_i = c_i R_i$  and  $V_j = c_j R_j$ , is not convincing. If arbitrary value transformations of reinforcement are permissible, then what prevents us from deriving the power law from power function value transformations alone:  $B_i/B_j = V_i/V_j = c_i R_i^s / c_j R_j^s = (c_i/c_j)(R_i/R_j)^s$ ?

These objections notwithstanding, Allen's proof shows that a relatively general constraint on lawful response-reinforcement relationships (Equation 3) is in fact closely related to an apparently more specific one (Equation 1). In this paper, I take this strategy one step further

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and derive the conditions under which a more general class of behavioral laws reduces to the power law.

### THE UNDERLYING ASSUMPTIONS

Consider the following constraints on behavioral laws pertaining to concurrent-schedule experiments with  $n \geq 2$  alternatives, where alternatives refer to specific reinforcer-schedule-operandum combinations. Thus, alternative  $i$  may designate a VI schedule without COD, delivering 3 s of food for pecks on a red key. (Changing the scheduled rate of reinforcement does not constitute a change in alternatives but rather a change in the rate of reinforcement for the same alternative.)

(A1) *Controlling variables: Response rates are continuous, differentiable functions of the rates of reinforcement obtained in the experiment,*

$$B_i = F_i(R_1, \dots, R_n), i = 1, \dots, n,$$

and only reinforcement can maintain responding:

$$R_i = 0 \text{ implies } B_i = 0.$$

This assumption ensures that we can express response rates in all experiments by means of a single set of functions,  $F_i(R_1, \dots, R_n)$ , with the proviso that the absence of any alternative  $i$  from a given experiment is indicated by setting  $R_i$  equal to zero.

Next, I define for each pair of alternatives,  $i$  and  $j$ , a *relative preference function*,  $P_{ij}(R_1, \dots, R_n)$ :

$$\frac{B_i}{B_j} = P_{ij}(R_1, \dots, R_n) = \frac{F_i(R_1, \dots, R_n)}{F_j(R_1, \dots, R_n)}. \quad (4)$$

(A2) *Relative homogeneity: Relative preference is not affected by proportional increases ( $\lambda > 1$ ) or decreases ( $\lambda < 1$ ) in all reinforcement rates:*

$$P_{ij}(\lambda R_1, \dots, \lambda R_n) = P_{ij}(R_1, \dots, R_n),$$

for all alternatives  $i, j$ , and all values of  $\lambda$  greater than zero. In other words, preference, as measured by relative rates of response, does not depend on the overall rate of reinforcement,  $\Sigma R_i$ , but only on the way in which this total is distributed among the alternatives.

I now introduce the third and final assumption by means of an intermediate condition

which, it turns out, is necessary for deriving the power law. We will say that reinforcement rates can be scaled independently of context if there exists a set of functions  $V_i(R_i)$ , one for each alternative  $i$ , such that relative response rate for two alternatives is equal to the relative scaled value of their reinforcement rates:

$$\frac{B_i}{B_j} = P_{ij}(R_1, \dots, R_n) = \frac{V_i(R_i)}{V_j(R_j)}. \quad (5)$$

Regardless of how one interprets the functions  $V_i$ , two key properties of Equation 5 stand out. First, all reinforcement rates other than  $R_i$  and  $R_j$  drop out of the relative preference function, and second, what remains of that function can be additively decomposed into a part depending only on  $R_i$  and a part depending only on  $R_j$ :

$$\log P_{ij}(R_1, \dots, R_n) = \log V_i(R_i) - \log V_j(R_j).$$

The relative preference functions that satisfy this condition (Equation 5) can be characterized in two different ways. The first formulation, given by Assumption A3, is conceptually more transparent but has a drawback in that it applies only to experiments with three or more alternatives.

(A3) *Relative independence: Relative preference is not affected by the rate of reinforcement for a third alternative,*

$$\frac{\partial}{\partial R_k} P_{ij}(R_1, \dots, R_n) = 0,$$

for all distinct alternatives  $i, j, k$ .

One can think of this assumption as a distant relative of the independence-from-irrelevant-alternatives principle (Luce, 1959), because it asserts that preference does not depend on reinforcement for competing alternatives. Luce's choice axiom is actually much stronger than A3, which maintains that preference  $P_{ij}$  does not depend on the availability of a third alternative only as long as the obtained rates of reinforcement for  $i$  and  $j$  are held constant; the choice axiom makes no such qualification (see Prelec, 1982, pp. 206-207; Prelec & Herrnstein, 1978).

As a direct consequence of Assumption A3, we can write relative preference as a function of only two arguments:

$$P_{ij}(R_1, \dots, R_n) = F_{ij}(R_i, R_j). \quad (6)$$

It is obvious that Equation 5 implies Equation

6; what is less self-evident is that the converse implication also holds, provided there are at least three alternatives. The proof is relatively straightforward. Because of the tautology,  $(B_i/B_k) = (B_i/B_j)(B_j/B_k)$ , relative preference functions must obey the identity:

$$F_{ik}(R_i, R_k) = F_{ij}(R_i, R_j)F_{jk}(R_j, R_k). \quad (7)$$

Equation 7 is a form of Sincov's functional equation (Aczel, 1966), usually written as  $F(x, z) = G(x, y) + H(y, z)$ .

Appendix I shows that the solutions to Equation 7 are all versions of Equation 5:

$$\begin{aligned} F_{ij}(R_i, R_j) &= \frac{V_i(R_i)}{V_j(R_j)}, \\ F_{ik}(R_i, R_k) &= \frac{V_i(R_i)}{V_k(R_k)}, \\ F_{jk}(R_j, R_k) &= \frac{V_j(R_j)}{V_k(R_k)}. \end{aligned} \quad (8)$$

A different formulation of independence (Equation 5) is given below.

(B3) *Additive independence: A change in reinforcement rate for alternative i, from  $R_i$  to  $R_i^*$ , changes the relative preference for i over every other alternative by the same proportion, irrespective of the rates of reinforcement for other alternatives:*

$$\begin{aligned} &\frac{P_{ij}(R_1, \dots, R_i, \dots, R_n)}{P_{ij}(R_1, \dots, R_i^*, \dots, R_n)} \\ &= \frac{P_{ik}(R_1', \dots, R_i, \dots, R_n')}{P_{ik}(R_1', \dots, R_i^*, \dots, R_n')} \end{aligned}$$

for all alternatives  $j, k$  distinct from  $i$  (including  $j = k$ ).

As will become apparent in the next section, the sole reason for including B3 along with A3 is to be able to deal with the important special case when there are only two alternatives; for three or more alternatives, Assumptions A3 and B3 are completely equivalent. To prove this, notice first that Equation 5 implies B3, since both sides of the equality in B3 are then equal to  $V_i(R_i)/V_i(R_i^*)$ . Therefore, A3 implies B3 as well. Conversely, using the fact that  $P_{ik} = 1/P_{ki}$ , we can rewrite B3 as

$$\begin{aligned} P_{ki}(R'_1, \dots, R_i, \dots, R'_n)P_{ij}(R_1, \dots, R_i, \dots, R_n) &= \\ P_{ki}(R'_1, \dots, R_i^*, \dots, R'_n)P_{ij}(R_1, \dots, R_i^*, \dots, R_n). \end{aligned}$$

Now let  $R'_1 = R_1, \dots, R'_n = R_n$ , and apply the

identity  $P_{ki}P_{ij} = P_{kj}$  (viz. Equation 7) to obtain an equation,

$$P_{kj}(R_1, \dots, R_i, \dots, R_n) = P_{kj}(R_1, \dots, R_i^*, \dots, R_n),$$

which demonstrates that relative preference for any two alternatives  $k, j$  does not depend on the rate of reinforcement for a third alternative,  $i$ .

For experiments with two alternatives, Assumption B3 permits a direct derivation of the value functions in Equation 5. Let  $R_1^\circ$  and  $R_2^\circ$  be any pair of reinforcement rates that generates indifference:

$$P_{12}(R_1^\circ, R_2^\circ) = 1,$$

and define the values of  $R_1$  and  $R_2$  as

$$V_1(R_1) = P_{12}(R_1, R_2^\circ),$$

and

$$V_2(R_2) = P_{21}(R_1^\circ, R_2).$$

Then, applying B3, we have

$$\begin{aligned} P_{12}(R_1, R_2) &= \frac{P_{12}(R_1^\circ, R_2)P_{12}(R_1, R_2^\circ)}{P_{12}(R_1^\circ, R_2^\circ)} \\ &= \frac{V_1(R_1)}{P_{21}(R_1^\circ, R_2)P_{12}(R_1^\circ, R_2^\circ)} \\ &= \frac{V_1(R_1)}{V_2(R_2)}, \end{aligned}$$

as required.

## DERIVATION OF THE POWER LAW

I start with Assumptions A1, A2, and A3 because they lead to a proof that is closely related to that of Allen (1981). Again, we have to assume that there are at least three alternatives; otherwise A3 has no force. By relative independence, we are permitted to express the response ratio,  $B_i/B_j$ , as a function of  $R_i$  and  $R_j$  (viz. Equation 6). Applying relative homogeneity to Equation 6, and letting  $\lambda = 1/R_j$ , shows, furthermore, that relative preference is a function of the ratio,  $R_i/R_j$ :

$$\begin{aligned} F_{ij}(R_i, R_j) &= F_{ij}(\lambda R_i, \lambda R_j) \\ &= F_{ij}(R_i/R_j, 1) \\ &= f_{ij}(R_i/R_j), \text{ for all } 1 \leq i, j \leq n. \end{aligned} \quad (9)$$

By substituting the functions  $f_{ij}$ ,  $f_{ik}$ , and  $f_{jk}$  into Equation 7, we obtain a version of Cauchy's functional equation:

$$f_{ik}(R_i/R_k) = f_{ij}(R_i/R_j)f_{jk}(R_j/R_k), \quad (10)$$

commonly written as

$$f(xy) = g(x)h(y) \quad (11)$$

(note:  $x = R_i/R_j$ ,  $y = R_j/R_k$ ).

If Equation 11 holds for all positive real numbers,  $x, y$ , then the functions  $f, g$ , and  $h$  are power functions with common exponent  $s$  but distinct multiplicative constants,  $a, b$ , and  $ab$  (see Appendix II):  $f(x) = ab x^s$ ,  $g(x) = a x^s$ ,  $h(x) = b x^s$ .

Applying this set of solutions to  $f_{ij}$  in Equation 10 yields a power law with bias coefficient  $c_{ij}$ :

$$\frac{B_i}{B_j} = f_{ij}(R_i/R_j) = c_{ij} \left( \frac{R_i}{R_j} \right)^s.$$

The bias coefficients can be different for different pairs of alternatives, but they are mutually constrained by the identity:

$$c_{ij} = c_{ik}c_{kj}.$$

In his derivation, Allen started with the assumption that all relative preference functions are identical,  $f_{ij} = f_{jk}$ , etc. This leads to a stricter version of the Cauchy equation,  $f(xy) = f(x)f(y)$ , whose solutions are power functions without a bias parameter. The only consequence of permitting each alternative pair to have a distinct relative preference function is to attach a bias parameter to the power law.

This proof, like Allen's, requires that there be at least three concurrently available alternatives, which makes it irrelevant to most empirical work on the power law. If, however, we substitute Assumption B3 for Assumption A3 in the starting list of assumptions, then we can construct a proof that works for the case  $n = 2$ , as well. Putting together Equations 5 and 9 yields:

$$f_{ij}(R_i/R_j) = \frac{V_i(R_i)}{V_j(R_j)}.$$

(Recall that the right side of this equation follows from B3 and the left from A2.) Substituting the functions

$$f(x) = f_{ij}(x),$$

$$g(x) = V_i(x),$$

$$h(x) = V_j(1/x)^{-1},$$

leaves us with the general Cauchy equation (11),

$$f(R_i/R_j) = g(R_i)h(1/R_j),$$

whose solution, we already know, is the general power law, Equation 1.

## THE TWO TYPES OF (THEORETICAL) DEVIATION FROM THE POWER LAW: INHOMOGENEITY AND NON-INDEPENDENCE

So far we have established that the power law is the only behavioral law (having reinforcement rates as controlling variables) characterized by the properties of relative homogeneity and independence. What does this mean? The empirical interpretation of these two assumptions is hindered by an ambiguity—one has to decide whether to include unscheduled (and therefore unobserved) alternatives in the total list of present alternatives (Herrnstein, 1970). If one believes that no unscheduled reinforcers exist, then Assumptions A2 and A3 have clear meaning: An experiment in which all rates of reinforcement are proportionally varied constitutes a direct test of relative homogeneity, while a direct test of relative independence can be arranged with experiments containing three or more alternatives. If, on the other hand, one assumes that there do exist contaminating sources of unscheduled reinforcement, then a proportional increase in reinforcement rates in concurrent-schedule experiments becomes a test of independence, rather than homogeneity, because it constitutes an indirect manipulation of the rate of unscheduled reinforcement relative to the rate of schedule-delivered reinforcement. The notion of relative homogeneity, however, is now robbed of its operational definition; it is difficult to conceive of an experiment that would proportionally increase scheduled and unscheduled (i.e., background) received reinforcement rates.

We have to be clear here about what it is that we wish to derive. If, along with Allen (1982), we wish to derive the *unrestricted* power law—the proposition that power functions hold for all possible alternative pairs, including unscheduled ones—then we have to assume unrestricted homogeneity and independence (Assumptions A2 and A3). The ambiguity in the assumptions is then balanced by the ambiguity in the conclusions, for what em-

irical significance can be attached, for example, to the statement that

$$\frac{B_1}{B_0} = c \left( \frac{R_1}{R_0} \right)^s,$$

where  $B_0$  and  $R_0$  stand for unscheduled, and therefore unobserved, response and reinforcement rates?

Fortunately, it is possible to maintain an agnostic position about background reinforcers and still preserve the power law derivation. The only thing one needs to assume is that unscheduled reinforcement, if it exists, does not affect relative preference between observable alternatives; that is,

$$\frac{\partial P_{ij}}{\partial R_0} = 0,$$

which is plausible enough. Once relative preference is protected against variation in background reinforcement, then it no longer matters (for deriving the power law between *observed* alternatives, that is) whether or not background reinforcers are included in the list  $R_1, \dots, R_n$ .

The empirical power law for the case  $n = 2$  now stands or falls with two entirely verifiable propositions. Homogeneity implies that relative response rate is some determinate function of relative reinforcement rate,

$$\frac{B_1}{B_2} = f_{12} \left( \frac{R_1}{R_2} \right),$$

and independence (B3) implies that log response ratio is an additive function of reinforcement rates,

$$\log \frac{B_1}{B_2} = \log V_1(R_1) - \log V_2(R_2).$$

Given sufficient variation in reinforcement rates, failure of the power law indicates failure of at least one of these two primitive properties.

What sort of experimental conditions might promote selective violation of homogeneity or independence? Using qualitatively different reinforcers is one possibility. Let us for the moment borrow the terminology of economics and classify reinforcers along a dimension of necessity/luxury, so that reinforcers that are prepotent when the overall rate of reinforcement is high define the "luxury" end of the continuum, and those that are prepotent in lean experimental environments define the

"necessity" end. In other words, the reinforcing power of necessities diminishes substantially with increases in absolute reinforcement rate, while that of luxuries remains steady. Now, homogeneity will not obtain unless the two types of reinforcers delivered by the concurrent schedules fall roughly in the same region on this continuum. This can be clearly illustrated by generalizing the general matching law so as to allow for different exponents on the two reinforcement rates:

$$\frac{B_1}{B_2} = c_{12} \left( \frac{R_1^{s_1}}{R_2^{s_2}} \right). \quad (12)$$

This equation satisfies independence but does not satisfy homogeneity, unless the two exponents are equal:

$$P_{12}(\lambda R_1, \lambda R_2) = \lambda^{(s_1 - s_2)} P_{12}(R_1, R_2).$$

The exponents,  $s_1$  and  $s_2$ , measure how fast the effectiveness of reinforcement decreases with reinforcement rate and are therefore crude indices of luxury/necessity. It would be worthwhile in future fitting of the power law to start first with a linear regression of Equation 12,

$$\log(B_1/B_2) = s_1 \log R_1 - s_2 \log R_2 + \log c_{12},$$

and then proceed with a statistical test of the homogeneity condition,  $s_1 = s_2$ .

In order to produce violations of independence, we would have to find reinforcers that interact in interesting ways, such as, for example, salty and bland foods. With this combination, it is conceivable that the relationship between relative preference ( $B_1/B_2$ ) and relative consumption ( $R_1/R_2$ ) is bitonic—increasing when relative consumption of salty food ( $R_1$ ) is low and decreasing when it is high.

The derivation presented in this paper does not satisfy our curiosity about one very important point, which is why the power law fits such a large body of data. What it does, instead, is push the issue one level back and makes us wonder, in turn, why homogeneity and independence should hold for these same data. The answer may have as much to do with the way we design experiments as with the behavioral principles that are the object of our study. Relative independence (A3), for example, is virtually a normative criterion for a good choice experiment; certainly, if I wish to measure preference between two alternatives,  $i$  and  $j$ , I will avoid procedures in which preference is sensitive to the presence of a

third factor,  $k$ . Therefore, in trying to evaluate the significance of any formal derivation of the power law, we should keep in mind that the power law reflects an interaction between the subjects' behavior and those semi-articulated principles that guide our selection of particular reinforcers and procedures over others.

### REFERENCES

- Aczel, J. (1966). *Lectures on functional equations and their applications*. New York: Academic Press.
- Allen, C. M. (1981). On the exponent in the "generalized" matching equation. *Journal of the Experimental Analysis of Behavior*, 35, 125-127.
- Allen, C. M. (1982). Comment on Houston's arguments. *Journal of the Experimental Analysis of Behavior*, 38, 113-114.
- Baum, W. M. (1974). On two types of deviation from the matching law: Bias and undermatching. *Journal of the Experimental Analysis of Behavior*, 22, 231-242.
- Baum, W. M. (1979). Matching, undermatching, and overmatching in studies of choice. *Journal of the Experimental Analysis of Behavior*, 32, 269-281.
- Herrnstein, R. J. (1961). Relative and absolute strength of response as a function of frequency of reinforcement. *Journal of the Experimental Analysis of Behavior*, 4, 267-272.
- Herrnstein, R. J. (1970). On the law of effect. *Journal of the Experimental Analysis of Behavior*, 13, 243-266.
- Houston, A. I. (1982). The case against Allen's generalization of the matching law. *Journal of the Experimental Analysis of Behavior*, 38, 109-111.
- Luce, R. D. (1959). *Individual choice behavior: A theoretical analysis*. New York: Wiley.
- Prelec, D. (1982). Matching, maximizing, and the hyperbolic reinforcement feedback function. *Psychological Review*, 89, 189-230.
- Prelec, D., & Herrnstein, R. J. (1978). Feedback functions for reinforcement: A paradigmatic experiment. *Animal Learning & Behavior*, 6, 181-186.
- Staddon, J. E. R. (1968). Spaced responding and choice: A preliminary analysis. *Journal of the Experimental Analysis of Behavior*, 11, 669-682.

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## APPENDIX I

*Solution of Sincov's functional equation (after Aczel, 1966, p. 303). Let*

$$F(x, z) = G(x, y)H(y, z),$$

and for some particular values  $x^0$  and  $z^0$  define:

$$h(z) = F(x^0, z),$$

$$g(y) = G(x^0, y),$$

$$f(x) = F(x^0, z^0)/F(x, z^0).$$

$$H(y, z) = \frac{F(x, z)}{G(x, y)} = \frac{F(x^0, z)}{G(x^0, y)} = \frac{h(z)}{g(y)},$$

$$G(x, y) = \frac{F(x, z^0)}{H(y, z^0)} = \frac{F(x, z^0)g(y)}{h(z^0)} = \frac{g(y)}{f(x)},$$

$$F(x, z) = \frac{G(x, y)}{H(y, z)} = \frac{g(y)h(z)}{f(x)g(y)} = \frac{h(z)}{f(x)},$$

as required.

## APPENDIX II

*Solution to the general Cauchy equation. Starting with*

$$f(xy) = g(x)g(y),$$

we substitute

$$x' = \log x$$

$$y' = \log y,$$

and

$$F(z) = \log f(e^z),$$

$$G(z) = \log g(e^z),$$

$$H(z) = \log h(e^z),$$

to obtain

$$F(x' + y') = G(x') + H(y').$$

If we let  $G(0) = a$ , and  $H(0) = b$ , then

$$G(x') = F(x') - b,$$

$$H(y') = F(y') - a,$$

allowing us to write

$$F(x' + y') = F(x') + F(y') - a - b.$$

Substituting for  $Q(z) = F(z) - a - b$ , we get a functional equation,

$$Q(x' + y') = Q(x') + Q(y'),$$

whose solution (Allen, 1981) is

$$Q(x) = sx.$$

This implies that the solution to  $F(x)$  is

$$F(x) = sx + a + b.$$

(\*) Substituting back into (\*) yields the power function:

$$\begin{aligned} f(x) &= e^{F(\log x)} \\ &= e^{(s \log x + a + b)} \\ &= (e^a)(e^b)x^s. \end{aligned}$$

Similarly, we can show that

$$g(x) = (e^a)x^s,$$

and

$$h(x) = (e^b)x^s.$$